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Wake curvature and the Kutta condition

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The potential problem for the flow at high Reynolds numbers R outside the boundary layer and wake of a thin flat plate at small incidence with allowance for displacement thickness is not fully defined unless the position of the wake is known in advance. The Kutta–Joukowski hypothesis does not provide a satisfactory first approximation to this because of the singularity in curvature of the streamline springing from the trailing edge in inviscid flow, which implies that the initial curvature of the wake in the real flow will be large enough to cause a modification to the potential flow. The net vorticity per unit length in a curved wake is approximately $U_\infty \delta_{2\infty} d\theta_0/ds$, where U_∞, $\delta_{2\infty}$ and $d\theta_0/ds$ are respectively the undisturbed stream velocity, momentum thickness at infinity and curvature. The outer potential problem is set up with a vortex sheet of this strength to represent the wake, leading to a singular integro-differential equation for $\theta_0(s)$. From the particular solution we obtain a proportionate correction $-(C_D/4\pi)(\log 4/C_D)$ to the Kutta–Joukowski circulation, where C_D is the drag coefficient. For laminar flow this is of order $R^{-\frac{1}{2}}\log(1/R)$. However, the solution also contains an arbitrary constant which cannot be settled without an examination of the near wake. The recent work of Brown & Stewartson (1970) suggests that this may lead to a term of the lower order $R^{-\frac{3}{8}}$, but depends on the assumption, not supported by the present analysis, that the pressure rise across the wake is $o(R^{-\frac{1}{4}})$.

1. Introduction

The idea of calculating the outer field of flow past a two-dimensional obstacle at high Reynolds numbers as the inviscid flow past its displacement surface goes back to Prandtl. The underlying reasoning, especially for the three-dimensional case, was set out by Lighthill (1958). In recent years the two-dimensional calculation has derived some further mathematical justification from the method of matched asymptotic expansions, as being the second outer approximation to the solution of the full Navier–Stokes equations when $R \to \infty$, R being the Reynolds number based, say, on the chord. It is clearly appropriate only for attached flows, and therefore only for streamline shapes below the stall; and even for such shapes a separation bubble may be inescapable in the trailing-edge region. Moreover, the flow in the wake must be treated as steady (although possibly turbulent) if the mathematical problem is to be well-posed, but this may well be an oversimplification.

Van Dyke's (1964) demonstration that the outer flow is inviscid and irrotational up to order $R^{-\frac{1}{2}}$, with inner boundary condition that ψ at the surface equals the displacement flux, was carried out for a semi-infinite flat plate, but the same reasoning would give the same inner boundary condition in terms of the displacement thickness along the centre-line of the wake behind a finite plate at zero incidence. This would produce a term of order $R^{-\frac{1}{2}}$ in the outer stream function. In the immediate trailing-edge neighbourhood, however, it appears from the recent work of Stewartson (1969) and Messiter (1970) that the singularity in the slope of the displacement thickness may result in a lower-order term ($R^{-\frac{3}{8}}$) in the local outer flow. Brown & Stewartson (1970) extend these arguments to the case of asymmetric flow past a flat plate at incidence; and Riley & Stewartson (1969) have made a start on the flow near the rear separation point for a symmetrical wedge section of half angle α.

Some years ago the present author (Spence 1954), following the work of Preston (1949), carried out calculations of the flow past particular aerofoil sections with allowance for the displacement thickness both on the aerofoil thickness and on the wake centre-line. It was assumed that the gradient of the displacement thickness was finite at the trailing edge—an assumption now called into question by Stewartson—but the rapid decrease of displacement thickness immediately downstream of the trailing edge was certainly a conspicuous feature of the predicted flow. In these calculations the circulation round the aerofoil was calculated from a 'modified Joukowski' requirement that the pressure difference between points just outside the boundary layer immediately above and immediately below the trailing edge should be the same when calculated through the trailing-edge boundary layer (i.e. by allowing for the boundary-layer streamlines near the trailing edge) as when calculated from the external irrotational flow. This gave reasonable agreement with experiment, but is essentially empirical in the selection of the points at which consistency is to be imposed between inner and outer flows.

The correct asymmetrical outer problem is to find a stream function ψ satisfying $\nabla^2\psi = 0$ and appropriate conditions at infinity, and such that $\psi \rightarrow U\delta_1$ on the solid surface and on either side of the wake centre-line $\psi = 0$ say, where δ_1 is the appropriate displacement thickness, and U the local free-stream velocity. However, this boundary condition is insufficient to define the problem unless we know the position of the wake (as we do that of the surface) in advance. In the absence of this knowledge, a further boundary condition on ψ along the wake is needed to complete the specification of the problem, and the location of the wake will then emerge as part of the answer. The simplest such further boundary condition would come from applying the requirement of boundary-layer theory, that the pressure should not vary across the wake. This would hold provided $\kappa\delta$ were negligible to the approximation considered, where κ is the wake curvature and δ a representative thickness. In the zeroth approximation, i.e. that of inviscid flow with the Joukowski circulation, the wake centre-line would be the streamline springing from the trailing edge. This streamline however has infinite curvature, which gives rise to a singular perturbation problem as $\delta \rightarrow 0$. The problem was first noted in a paper by Spence & Beasley (1960) containing further calculations of the Preston type.

The aim of the present paper is to formulate and solve this problem. Essentially we have to balance the inner and outer flows so that the pressure rise across the wake is consistent with the streamline curvature which it induces. It turns out that this is possible if the curvature is of order $\alpha(c\delta)^{-\frac{1}{2}}$, where c and α are a reference length and flow inclination. The pressure rise across the wake is then of order $\alpha(\delta/c)^{\frac{1}{2}}\rho U_\infty^2$, i.e. for a laminar boundary layer, of order $\alpha R^{-\frac{1}{4}}\rho U_\infty^2$. Since the displacement thickness is of order $R^{-\frac{1}{2}}c$, the wake curvature is responsible for a lower-order effect if $\alpha R^{\frac{1}{4}} \gg 1$, but even if the two orders were comparable, the curvature would have to be calculated first in order to find the position of the centre-line for use in calculating the displacement effect on the outer flow. Only the boundary-value problem for the curvature will be treated here, and we consider only a flat plate aerofoil. We do not in fact use the Kutta–Joukowski hypothesis to provide a first approximation; instead we solve for the outer flow up to the order $\alpha(\delta/c)^{\frac{1}{2}}$ in one step, and the circulation is then found to have the Kutta–Joukowski value in the limit $\delta/c \to 0$.

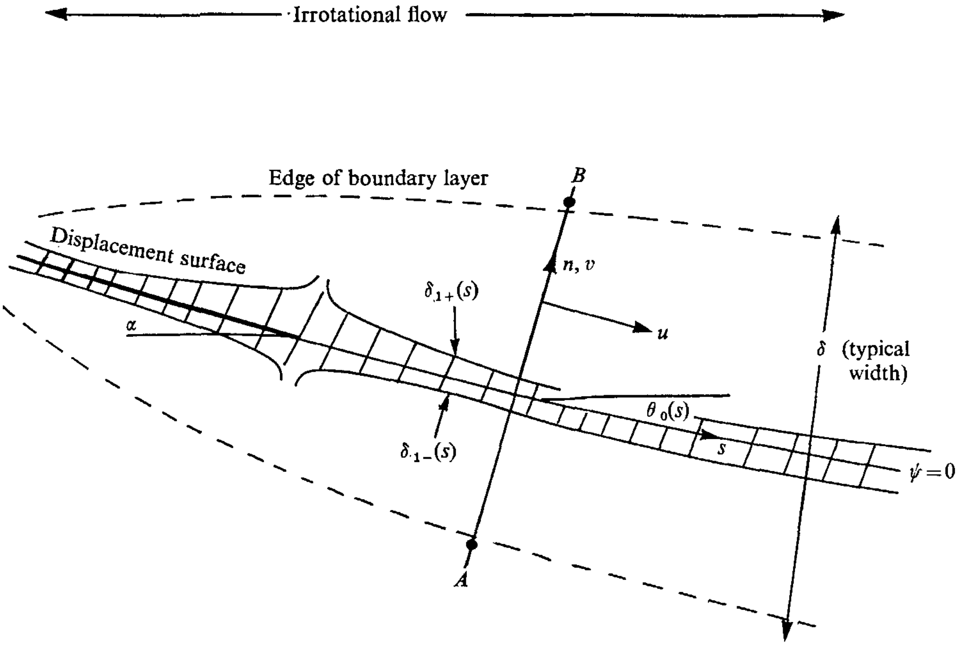

FIGURE 1. Wake streamlines and displacement surface (schematic).

In posing the problem it is more convenient to cast the analysis in terms of the net vorticity in a section of the wake than in terms of the pressure rise, because the latter depends on the precise points at which it is calculated, whereas the value of vorticity is uniquely determined, and is likewise proportional to $\kappa\delta$. The aerofoil and wake then appear to the outer flow as a semi-infinite vortex sheet located on a streamline $\psi = 0$, whose position downstream of the trailing edge is not known beforehand. The general scheme is indicated in figure 1.

In § 2 the Navier–Stokes equations are written in curvilinear co-ordinates to permit calculation of the vorticity. The turbulent terms are retained in this discussion, but it is argued that they do not affect the vorticity–curvature relationship, except indirectly through the momentum thickness δ_2 which is a constant of proportionality. In fact δ_2 varies slightly along the wake, but we may treat it as constant with an error of higher order than the effect considered, and related to the drag coefficient C_D by $\delta_2 = \frac{1}{2}cC_D$.

The potential problem for the outer flow is formulated in § 3, leading to an integro-differential equation for the slope $\theta_0(s)$ of the wake streamlines. A co-ordinate stretching transformation is found in § 4 which displays the balance between vorticity and curvature in both inner and outer flows when the curvature $d\theta_0/ds$ is of order $(c\delta_2)^{-\frac{1}{2}}$. The circulation round an infinite contour, which gives the lift on the wing, is found from the solution of the integral equation, and turns out to be $1-(C_D/4\pi)(\log 4/C_D)$ times the Kutta–Joukowski value $\pi\alpha U_\infty c$, so a proportional reduction in lift of order $R^{-\frac{1}{2}}\log R$ is produced by the effect; this is, in any case, of lower order than the contribution that would be produced in the same term by the displacement effect.

There is, however, a term with arbitrary coefficient A, say, in the solution, which cannot be fixed from 'outer' considerations. If A is of order unity, the term of order $C_D \log C_D$ is the dominant correction. This is probably true for turbulent motion, but in the case of laminar flow the arguments of Brown & Stewartson (1970) who have examined the 'inner' viscous flow close to the trailing edge strongly suggest that A is of order $R^{\frac{1}{8}}$, and is responsible for a term of the lower order $R^{-\frac{3}{8}}$ in the circulation defect. However they depend on the assumption that the pressure rise across the wake is zero to order $R^{-\frac{1}{4}}$, whereas we find it to be of order $AR^{-\frac{1}{4}}$, which would be consistent with their assumption only if $A = 0$. In that case $C_D \log C_D$ is the leading term.

2. Vorticity in a curved wake

Let s denote distance along the streamline $\psi = 0$ which springs from the trailing edge, and $\kappa(s)$ be the curvature of this streamline. Within the wake we choose a system of co-ordinates (s, n) such that s is constant on straight lines normal to the streamline $\psi = 0$, and n measures distance from $\psi = 0$ along such lines. Taking (u, v) as mean velocity components in s and n directions, we introduce a characteristic width δ say with which to scale distances and velocities *across* the wake, while making s non-dimensional with the aerofoil chord length c. Accordingly, in the full time-averaged Navier–Stokes equations for two-dimensional incompressible flow, write

$$\left.\begin{aligned} &s = cX, \quad n = \delta Y, \quad u = U_\infty U, \quad v = (\delta/c)\,U_\infty V, \quad \zeta = (U_\infty/\delta)\,Z, \\ &p = \rho U_\infty^2 P, \quad -\overline{u'v'} = (\delta U_\infty^2/c)\,T, \quad \kappa = K/\delta, \quad H = 1-KY. \end{aligned}\right\} \tag{2.1}$$

(Here p, ρ, ζ are pressure, density and vorticity, U_∞ the undisturbed stream velocity, and $-\overline{u'v'}$ the Reynolds stress.) With the exclusion of terms that are formally of order δ/c, the equations become

$$U\frac{\partial U}{\partial X} + V\frac{\partial}{\partial Y}(HU) + \frac{\partial P}{\partial X} = -\frac{1}{R}\left(\frac{c}{\delta}\right)^2\left[\frac{\partial}{\partial Y}(HZ) + KZ\right] + \frac{\partial}{\partial Y}(HT) - KT, \tag{2.2}$$

$$\frac{KU^2}{H} + \frac{\partial P}{\partial Y} = 0, \tag{2.3}$$

where
$$-\frac{1}{H}\frac{\partial}{\partial Y}(HU) = Z \tag{2.4}$$

is the dimensionless vorticity, and $R = U_\infty c/\nu$. The continuity equation is exactly

$$\frac{\partial U}{\partial X}+\frac{\partial}{\partial Y}(HV) = 0. \tag{2.5}$$

For laminar flow $T = 0$ and we should take $\delta/c = R^{-\frac{1}{2}}$, thus making the coefficient of the viscous term unity in (2.2). For turbulent flow the coefficient would be a negative power of R, about $R^{-\frac{3}{5}}$, and the viscous term can be omitted. If the physical curvature κ is of order c^{-1}, K is of order δ/c and may also be excluded. Then $H = 1$ and the equations are just those for the boundary layer on a plane wall. In the case to be discussed, however, it turns out that there are regions in which κ is of order $(c\delta)^{-\frac{1}{2}}$; K is therefore of order $(\delta/c)^{\frac{1}{2}}$ and must be retained in a treatment that is accurate to this order. Terms of order K^2 can, however, be excluded, so (2.3) and (2.4) may be taken as

$$KU^2+\partial P/\partial Y = 0, \quad -\partial U/\partial Y+KU = Z, \tag{2.6}$$

respectively. Outside the wake, in the region of irrotational flow, Z and T vanish, and (2.2) reduces to Bernoulli's equation in the form $\partial(P+\frac{1}{2}U^2)/\partial X = 0$, with integral

$$P+\tfrac{1}{2}U^2 = \text{constant}, \tag{2.7}$$

in view of the upstream conditions.

Now let A and B be points just below and just above the wake on the same s line, as indicated in figure 1. The net vorticity per unit length at this station is then

$$\gamma(s) = \int_A^B \zeta dn = U_\infty \int_A^B Z\,dY = -U_\infty[U]_A^B+KU_\infty\int_A^B U\,dY$$

by (2.6). The difference in velocity across the wake can also be written in terms of the pressure difference

$$[P]_A^B = [-\tfrac{1}{2}U^2]_A^B = -\overline{U}[U]_A^B,$$

where $\overline{U} = \frac{1}{2}(U_A+U_B)$, by (2.7). Also by integrating the first equation of (2.6) across the wake we find

$$[P]_A^B = -K\int_A^B U^2 dY.$$

Elimination of $[P]_A^B$ and $[U]_A^B$ from the last three equations gives

$$\gamma = KU_\infty\int_A^B U[1-(U/\overline{U})]\,dY. \tag{2.8}$$

This expression is independent of the precise locations of A and B on the same s line.

In terms of the physical variables, the right-hand side is $\kappa_1\bar{u}\delta_2$, where

$$\delta_2(s) = \int_A^B \frac{u}{\bar{u}}\left(1-\frac{u}{\bar{u}}\right)dy \tag{2.9}$$

is the displacement thickness, and $\bar{u} = U_\infty \bar{U}$. (There is a slight imprecision in this definition, in that $\bar{u}\delta_2$ but not δ_2 itself is independent of the choice of A and B.) Now δ_2 satisfies a momentum integral equation

$$\frac{d\delta_2}{ds} + (\delta_2 + 2\delta_1)\frac{1}{\bar{u}}\frac{d\bar{u}}{ds} = 0, \tag{2.10}$$

with an error of order $\kappa^2\delta^3$, where δ_1 is the displacement thickness similarly defined.

From this it follows that the variation of δ_2 along the wake is of the same order as that of $\bar{u}$, and may be neglected in a first-order theory. Accordingly we set

$$\gamma(s) = U_\infty \delta_{2\infty} \kappa(s) = 2\mu c U_\infty \kappa(s), \tag{2.11}$$

say, where $\delta_{2\infty}$ is the value of δ_2 far downstream, which equals $\frac{1}{2}cC_D$, and

$$\mu = \tfrac{1}{2}\delta_{2\infty}/c = \tfrac{1}{4}C_D. \tag{2.12}$$

3. The potential problem for the outer flow

On the scale of the outer flow, $y/\delta \gg 1$. Van Dyke (1964) shows by expansion of the Navier–Stokes equations we then have $\nabla^2\psi = -\omega(\psi)$ up to order δ (in his notation $R^{-\frac{1}{2}}$), where ω is the vorticity. In the case he discusses of a semi-infinite flat plate at zero incidence $\omega(\psi)$ is zero everywhere, since it is zero upstream; but in the present case there is a vortex sheet of strength $\gamma(s)$ per unit length extending from the front stagnation point to infinity downstream along the streamline $\psi = 0$.

The complex velocity $w = qe^{-i\theta}$ due to the vortex distribution is given by

$$w(z) = \frac{1}{2\pi i}\int_{\psi=0} \frac{\gamma(t)\,dt}{t-z}, \tag{3.1}$$

where $ds = |ds|\,e^{i\theta_0}$ is an element along the streamline whose slope is $\theta_0(s)$, say. The values of w immediately above and immediately below a point $z = s$ of the streamline are connected by the Plemelj formula

$$\tfrac{1}{2}\{w^+(s) + w^-(s)\} = \frac{1}{2\pi i}\int_{\psi=0} \frac{\gamma(t)\,dt}{t-s}. \tag{3.2}$$

These values must equal the inner limits as $y \to y(s, \psi_0)$ from above and below of the outer flow, namely

$$w_\pm = q_\pm \exp\left[-i(\theta_0 + \delta_{1\pm})\right], \tag{3.3}$$

where $\delta_{1\pm}$ are the values of displacement thickness on either side. The effect of displacement thickness, like that of aerofoil thickness, is purely additive to the present approximation, and for the reasons mentioned in § 1 we do not consider it here. If now α is a representative scale for θ, then since $w^\pm = U_\infty(1 + O(\alpha))$ and $ds_1 = |ds_1|\,(1 + O(\alpha))$, with the exclusion of terms of order α^2 we can write the imaginary part of (3.2) as

$$U_\infty \theta_0(s) = \frac{1}{2\pi}\int_0^\infty \frac{\gamma(t)\,dt}{t-s}. \tag{3.4}$$

This is the standard integral equation of thin aerofoil theory, now derived in terms of distance along the streamline $\psi = 0$, which includes the surface of the plate as the interval $0 < s < c$. The boundary conditions to be applied are of mixed type: (i) On the interval representing the aerofoil

$$0 < s < c: \quad \theta_0(s) = -\alpha. \tag{3.5}$$

(ii) Along the wake, by (2.11)

$$s > c: \quad \gamma(s) = 2\mu c U_\infty \kappa(s). \tag{3.6}$$

These boundary conditions are the same as those satisfied by the slope and vorticity on a jet-flapped wing, which have been considered by Spence (1956, 1961), and others, except that the sign of the curvature term in (3.6) is different, corresponding to a momentum defect in the wake, in contrast to the momentum excess in a jet sheet. The equations can be combined to give a single integro-differential equation for the unknown slope $\theta_0(s)$ on $s > c$, as follows. First, invert (3.4) to give

$$\frac{\gamma(s)}{2U_\infty} = \frac{1}{\pi}\int_0^\infty \left(\frac{t}{s}\right)^{\frac{1}{2}} \frac{\theta_0(t)\,dt}{t-s}. \tag{3.7}$$

This holds on the whole interval $0 < s < \infty$. For $0 < t < c$ we can substitute $\theta_0(t) = -\alpha$ in the integrand, in accordance with (3.5). When the resulting integral is evaluated for $c < s < \infty$, on which range $\gamma(s)$ is given by (3.6), the equation

$$\mu c \frac{d\theta_0}{ds} = \frac{1}{\pi}\left[\int_c^\infty \left(\frac{t}{s}\right)^{\frac{1}{2}} \frac{\theta_0(t)\,dt}{t-s} - \alpha\left\{2\left(\frac{c}{s}\right)^{\frac{1}{2}} - \log\left(\frac{s^{\frac{1}{2}}+c^{\frac{1}{2}}}{s^{\frac{1}{2}}-c^{\frac{1}{2}}}\right)\right\}\right] \tag{3.8}$$

is obtained. An equivalent form is

$$\mu c \frac{d\theta_0}{ds} = \frac{1}{\pi}\int_c^\infty \left(\frac{t}{s}\right)^{\frac{1}{2}} \frac{\theta_0(t) - \theta_0^*(t)}{t-s}\,dt, \tag{3.9}$$

where

$$\theta_0^*(s) = -\alpha\left[1 - \left(\frac{s-c}{s}\right)^{\frac{1}{2}}\right]. \tag{3.10}$$

$\theta_0^*(s)$ is the slope of the zero streamline according to classical potential theory, i.e. in the absence of a wake, and exhibits the square-root singularity in curvature at $s = c$ previously referred to. Clearly $\theta_0(s) = \theta_0^*(s)$ when $\delta_{2\infty} = 0$, but the singularity in behaviour when this limit is approached through small values of $\delta_{2\infty}$ is seen if we invert (3.9) on the interval (c, ∞) to give $\theta_0(s)$ as an integral of $\theta_0'(t)$:

$$\theta_0(s) + \frac{c}{\pi}\left(\frac{s-c}{s}\right)^{\frac{1}{2}} \int_c^\infty \left(\frac{t}{t-c}\right)^{\frac{1}{2}} \frac{\theta_0'(t)\,dt}{t-s} = \theta_0^*(s). \tag{3.11}$$

Here, the derivative $\theta_0^{*\prime}(t)$ cannot be inserted in the integrand to give the next approximation to $\theta_0(s)$, since the resulting integral would diverge at $t = c$.

The solution of these equations for $\theta_0(s)$ on $c < s < \infty$ is discussed in the next section. [The distribution of vorticity on the chord $0 < s < c$ can be found in terms of $\theta_0(s)$ by transformations similar to the above.]

$$0 < s < c: \gamma(s) = -2U_\infty\left(\frac{c-s}{s}\right)^{\frac{1}{2}}\left[\alpha - \frac{\delta_{2\infty}}{2\pi}\int_c^\infty \left(\frac{t}{t-c}\right)^{\frac{1}{2}} \frac{\theta_0'(t)\,dt}{t-s}\right]. \tag{3.12}$$

The total clockwise circulation in a circuit surrounding the aerofoil and cutting the wake at infinity is

$$\Gamma = \pi U_\infty \alpha c[1-2\mu C], \tag{3.13}$$

where

$$C = \frac{1}{\pi\alpha}\int_c^\infty \left(\frac{t}{t-c}\right)^{\frac{1}{2}} \theta_0'(t)\,dt.$$

The coefficient $\pi U_\infty \alpha c$ is the Kutta–Joukowski value for the circulation. Here again, direct substitution of $\theta_0^{*\prime}(t)$ in the integrand would produce a singularity, and it is necessary to look further into the solution of (3.11) before C can be calculated for small $\delta_{2\infty}/c$.

Taylor (1925) showed that the Blasius theorem $L = \rho U_\infty \Gamma$ still holds when a vertical wake is present, provided Γ is calculated round a circuit cutting the streamlines in the wake at right angles far downstream; this result can also be verified directly in the present case, since the lift is the sum of

$$-\rho U_\infty \int_0^c \gamma(s)\,ds,$$

which is the normal force on the aerofoil, and the vertical component of the skin friction on the aerofoil, namely $\alpha\rho U_\infty^2 \delta_{2\infty}$ in the present approximation. In view of (2.9), the sum of these two terms is $-\rho U_\infty \int_0^\infty \gamma(s)\,ds = \rho U_\infty \Gamma$.

4. Solution of the integro-differential equation

In (3.9), write

$$s/c = x+1, \quad \theta_0(s) - \theta_0^*(s) = \alpha\phi(x)/(x+1)^{\frac{1}{2}}, \tag{4.1}$$

when the equation becomes

$$\mu\left[\phi'(x) - \frac{\frac{1}{2}\phi(x)}{x+1} + \frac{\frac{1}{2}}{x^{\frac{1}{2}}(x+1)}\right] = \frac{1}{\pi}\int_0^\infty \frac{\phi(y)\,dy}{y-x} \tag{4.2}$$

and for smooth flow at the trailing edge $\phi(0) = 0$.

It does not seem possible to solve this equation in closed form, and attempts to treat it numerically have not up to now been successful. However, an approximation to the solution when μ is small can be found as follows. First, scale x and ϕ so the first and third terms on the left are formally of the same magnitude as the right-hand side. This is accomplished with the transformation

$$x = \mu X, \quad \phi(x) = \mu^{\frac{1}{2}} f(X), \tag{4.3}$$

when (4.2) becomes

$$(1+\mu X)\,Lf(X) + \tfrac{1}{2}X^{-\frac{1}{2}} = \tfrac{1}{2}\mu f(X), \tag{4.4}$$

where

$$Lf(X) \equiv f'(X) - \frac{1}{\pi}\int_0^\infty \frac{f(Y)\,dY}{Y-X}. \tag{4.5}$$

The first approximation for small μ would be to solve the equation obtained by setting $\mu = 0$ on both sides of (4.4); but a better approximation to the solution in the region where $\mu X \gg 1$ is obtained if we retain μ on the left-hand side, while still discarding it on the right. Subsequently we can verify that the function $f(X)$

so found is uniformly bounded on $0 < X < \infty$, which provides some justification for the exclusion of $\mu f(X)$ from (4.4).

Accordingly we treat the equation

$$Lf(X)+g(X) = 0, \tag{4.6}$$

where $g(X) = \frac{1}{2}X^{-\frac{1}{2}}(1+\mu X)^{-1}$.
The solution can be written

$$f(X) = f_0(X)+Af_1(X), \tag{4.7}$$

where $f_0(X)$ is a particular solution of (4.6), A an arbitrary constant, and $f_1(X)$ satisfies the homogeneous equation

$$Lf_1(X) = 0. \tag{4.8}$$

The solutions are found in terms of the Laplace transforms

$$\hat{f}_k(\xi) = \int_0^\infty e^{-\xi X} f_k(X)\,dX \quad (k = 0, 1), \tag{4.9}$$

$\hat{f}_0(\xi)$ satisfies the integral equation

$$\xi\hat{f}(\xi)+\frac{1}{\pi}\int_0^\infty \frac{\hat{f}(\eta)\,d\eta}{\eta-\xi}+\hat{g}(\xi) = 0, \tag{4.10}$$

and $\hat{f}_1(\xi)$ the same equation with $\hat{g}$ absent. These are in the form of Carlemann's singular equation, and can be solved by reduction to a Hilbert problem (see for example Tricomi 1957, pp. 188–197). The details will be given in a separate report. The final results are

$$\hat{f}_0(\xi) = \Phi(\xi)\frac{\exp\Psi(\xi)}{1+\xi^2}-\frac{\hat{g}(\xi)}{\xi}, \tag{4.11}$$

$$\hat{f}_1(\xi) = \frac{\exp\Psi(\xi)}{1+\xi^2}, \tag{4.12}$$

$$\left.\begin{aligned} \text{where} \qquad \Phi(\xi) &= \frac{1}{2\mu^{\frac{1}{2}}}\int_0^\infty \exp\left\{\Psi(t)-\frac{t}{\mu}\right\}\frac{dt}{t+\xi} \\ \text{and} \qquad \Psi(\xi) &= \frac{1}{\pi}\int_0^\infty \frac{(\cot^{-1}t)\,dt}{t+\xi} \quad (\tfrac{1}{2}\pi \geqslant \cot^{-1}t \geqslant 0). \end{aligned}\right\} \tag{4.13}$$

These expressions are regular in the region $-\pi < \arg\xi < \pi$, i.e. in the ξ plane cut along the negative real axis, except for poles at $\xi = \pm i$. Inversion of the Laplace transform gives contributions which for small μ tend to

$$-(\tfrac{1}{2}\pi)^{\frac{1}{2}}\cos(X-\tfrac{1}{8}\pi), \quad 2^{\frac{1}{2}}\sin(X-\tfrac{1}{8}\pi) \tag{4.14}$$

from the poles of $\hat{f}_0(\xi), \hat{f}_1(\xi)$ respectively. In addition there are loop integrals

$$\frac{1}{2\pi i}\int_{-\infty}^{(0+)} \hat{f}_k(\xi)\,e^{\xi X}\,d\xi \tag{4.15}$$

$(k = 0, 1)$ along contours passing inside the poles. These integrals can be put in real form by taking the contour as the two sides of the negative real axis and using

Plemelj formulae to connect the values of Ψ immediately above and immediately below this axis. In this way we obtain

$$f_1(X) = \frac{1}{\pi}\int_0^\infty \frac{e^{K(t)-tX}}{t^{\frac{1}{2}}(t^2+1)^{\frac{3}{2}}}\,dt + \text{contribution from poles}, \tag{4.16}$$

where
$$K(t) = \frac{1}{\pi}\int_0^\infty \frac{\log|t-u|\,du}{1+u^2} = O(t\log t) \quad \text{as} \quad t \to 0.$$

The limiting behaviour as $X \to \infty$ is found by writing $s = tX$, when the integral becomes

$$\frac{1}{\pi X^{\frac{1}{2}}}\int_0^\infty \frac{e^{K(s/X)-s}\,ds}{s^{\frac{1}{2}}(1+(s^2/X^2))^{\frac{3}{2}}} \sim \frac{1}{\pi X^{\frac{1}{2}}}\int_0^\infty \frac{e^{-s}\,ds}{s^{\frac{1}{2}}} = \frac{1}{(\pi X)^{\frac{1}{2}}}. \tag{4.17}$$

Similar but more intricate analysis shows that the loop integral for $f_0(X)$ behaves like

$$\frac{1}{\mu^{\frac{1}{2}}}\left[\tan^{-1}\left(\frac{1}{\mu X^{\frac{1}{2}}}\right) - \tan^{-1}\left(\frac{1-\mu^*}{\mu X+\mu^*}\right)^{\frac{1}{2}}\right],$$

where $\mu^* = (\mu/\pi)(\log 1/\mu)$, as $\mu \to 0$ for fixed X.
This expression

$$\sim \frac{1}{2\pi X^{\frac{1}{2}}}(\log 1/\mu) \quad \text{as} \quad X \to \infty. \tag{4.18}$$

In passing, the behaviour of f_0 and f_1 near $X = 0$ may be noted: it is

$$f_0(X) = -X^{\frac{1}{2}} + O(X^2), \quad f_1(X) = X + O(X^2). \tag{4.19}$$

No restriction has been imposed on the values of $f_0(0)$, $f_1(0)$ but it appears that the above, both of which vanish at $X = 0$ are the only possible solutions. These results are sufficient to permit the calculation of the quantity C in (3.13). With the notation of (4.4), (4.6) and omission of the term that is uniformly of order μ,

$$C = \frac{1}{\pi}\int_0^\infty \{f'(Y)+g(Y)\}\,Y^{-\frac{1}{2}}\,dY. \tag{4.20}$$

Now equation (4.6) can be inverted to the form

$$f(X) + \frac{1}{\pi}\int_0^\infty \left(\frac{X}{Y}\right)^{\frac{1}{2}}\{f'(Y)+g(Y)\}\frac{dY}{Y-X} = 0. \tag{4.21}$$

(This can also be derived directly from (4.11).) The solution just quoted is of the form

$$f(X) = B\,e^{iX} + f^*(X), \tag{4.22}$$

say, where $f^*(X) = O(1/X^{\frac{1}{2}})$ as $X \to \infty$, and we now show that

$$C = \lim_{x\to\infty} X^{\frac{1}{2}} f^*(X). \tag{4.23}$$

To show this, substitute (4.22) into (4.21). Then since

$$e^{iX} + \frac{i}{\pi}\int_0^\infty \left(\frac{X}{Y}\right)^{\frac{1}{2}}\frac{e^{iY}\,dY}{Y-X} = \frac{1}{\pi}\int_0^\infty \left(\frac{X}{iV}\right)^{\frac{1}{2}}\frac{e^{-V}\,dV}{X-iV}, \tag{4.24}$$

we have

$$\begin{aligned} f^*(X) + \frac{1}{\pi}\int_0^\infty \left(\frac{X}{Y}\right)^{\frac{1}{2}}\{f^{*\prime}(Y)+g(Y)\}\frac{dY}{Y-X} &= -\frac{B}{\pi}\int_0^\infty \left(\frac{X}{iV}\right)^{\frac{1}{2}}\frac{e^{-V}\,dV}{X-iV} \\ &\sim -\frac{B}{X^{\frac{1}{2}}}\cdot\frac{1}{\pi}\int_0^\infty (iV)^{-\frac{1}{2}}e^{-V}\,dV, \end{aligned}$$

as $X \rightarrow \infty$. Therefore

$$\lim_{X\rightarrow\infty} X^{\frac{1}{2}}f^*(X) = \frac{1}{\pi}\int_0^\infty \{f^{*\prime}(Y)+g(Y)\}\frac{dY}{Y^{\frac{1}{2}}}-\frac{B}{(i\pi)^{\frac{1}{2}}}, \tag{4.25}$$

and substitution of (4.22) into (4.20) shows that the right-hand sides of (4.20) and (4.25) are the same.

From (4.17), (4.18) it therefore follows that

$$C = (1/2\pi)\log(1/\mu)+A/\pi^{\frac{1}{2}}. \tag{4.26}$$

5. Discussion

The present work was undertaken primarily to resolve the question of where the sources representing the displacement thickness of the wake should be located in a potential-flow calculation. This is answered in principle by the solution of § 4, which gives the function $\phi(x)$ and therefore the slope of the wake centre-line. It was not appreciated at the outset however that the solution would contain the arbitrary constant A which it appears can be fitted only by considering the details of the separating flow near the trailing edge.

The curvature at the trailing edge is proportional to A (as follows from (4.19)), so the assumption of zero pressure rise exactly at the trailing edge would require the value $A = 0$. The leading term in the circulation defect is then of order $R^{-\frac{1}{2}}\log R$ for laminar flow. Brown & Stewartson (1970), however, arrive at a term of order $R^{-\frac{3}{8}}$. In effect they assume a solution for $\phi(x)$ with asymptotic behaviour $\sim$ const$/X^{\frac{1}{2}}$, which is just that given by (4.17), and treat the pressures above and below the wake as equal. This would emerge from the present analysis if we adopt the scaling

$$x = \epsilon^3\tilde{x}, \quad \text{where} \quad \epsilon = R^{-\frac{1}{8}} \tag{5.1}$$

which Brown & Stewartson show is appropriate to the viscous region close to the trailing edge. Then since $\mu = O(\epsilon^4)$, (4.2) becomes

$$\frac{1}{\pi}\int_0^\infty \frac{\phi(\tilde{y})\,d\tilde{y}}{\tilde{y}-\tilde{x}} = O(\epsilon). \tag{5.2}$$

The solution of this equation with zero on the right-hand side is just

$$\phi(\tilde{x}) = \text{const}.\,\tilde{x}^{-\frac{1}{2}}. \tag{5.3}$$

To this extent their assumed outer solution is consistent with the present one, but the two approaches cannot be fully reconciled, since their final result would require our constant A to be of order ϵ^{-1}, and we should then predict a pressure rise of order ϵ across the wake, whereas in both analyses it is concluded that pressure variations are at most of order ϵ^2 (and Brown & Stewartson in fact set $\Delta p = 0$ downstream of the trailing edge).

Another unexpected feature of the solution is the oscillatory behaviour represented by the terms in $\substack{\sin\\ \cos}(X-\frac{1}{8}\pi)$ in (4.14). The oscillation decays in amplitude because of the factor $(x+1)^{-\frac{1}{2}}$ in (4.1). The mathematical reason for the oscillatory terms, and for the arbitrary constant in the solution, lies in the change in character of (4.2) when μ changes sign. For μ negative, i.e. a momentum excess, (4.2) is a form of the jet-flap integro-differential equation studied by Spence (1956, 1961) and possesses a one-parameter family of monotonically decaying solutions characterized by the value of $\phi(0)$. When μ is positive, however, the solution takes the form (4.22), and all solutions are such that $\phi(0) = 0$, and are instead characterized by the value of $\lim_{x \to 0} d(\phi(x) + x^{\frac{1}{2}})/dx$. The streamline curvature associated with these terms is of order $\alpha \delta_{2\infty}^{-\frac{1}{2}}$, and the strength of the corresponding vorticity distribution is therefore of order $\alpha U_\infty \delta^{\frac{1}{2}}$, i.e. $\alpha U_\infty R^{\frac{1}{4}}$, as stated in the introduction. The oscillatory terms do not however affect the overall expression for circulation, and their possible physical significance must remain an open question.

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